Immersion of transitive tournaments in digraphs with large minimum outdegree*

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Abstract

We prove the existence of a function $h(k)$ such that every simple digraph with minimum outdegree greater than $h(k)$ contains an immersion of the transitive tournament on k vertices. This solves a conjecture of Devos, McDonald, Mohar and Scheide.

In this note, all digraphs are without loops. Let D be a digraph. We denote by $V(D)$ its vertex set and $A(D)$ its arc set. A digraph D is *simple* if there is at most one arc from x to y for any $x, y \in V(D)$. Note that arcs in opposite directions are allowed. The *multiplicity* of a digraph D is the maximum number of parallel arcs in the same direction in D. For an arc $a = (u, v)$ of a digraph D, we say that u is the *tail* of a and v its *head*. The *outdegree* (resp. *indegree*) of a vertex v, denoted by $d^+(v)$ (resp. $d^-(v)$), is equal to the number of arcs a of D such that v is the tail (resp. head) of a. The *outneighbourhood* (resp. *inneighbourhood*) of a vertex v, denoted by $N^+(v)$ (resp. $N^-(v)$), is the set of vertices y such that (v, y) (resp. (y, v)) is an arc of D. A *directed path* P in a digraph D is a set of vertices $x_1, \ldots, x_k$ such that $(x_i, x_{i+1}) \in A(D)$ for all $1 \leq i \leq k-1$. A *directed cycle* C in a digraph D is a set of vertices $x_1, \ldots, x_k$ such that $(x_i, x_{i+1}) \in A(D)$ for all $1 \leq i \leq k-1$ and $(x_k, x_1) \in A(D)$. A digraph D is a *tournament* if exactly one of (u, v) and (v, u) is an arc of D for all distinct $u, v \in V(D)$. The *transitive tournament* on k vertices, denoted by TT_k, is the unique tournament on k vertices without any directed cycle. The *complete digraph* on k vertices is the simple digraph on k vertices with every possible arc.

We say that a digraph D contains an *immersion* of a digraph H if there exists a mapping such that vertices of H are mapped to distinct vertices of D, and the arcs of H are mapped to directed paths joining the corresponding pairs of vertices of D, in such a way that these paths are pairwise arc-disjoint. If the directed paths are pairwise internally vertex-disjoint, we say that D contains a *subdivision* of H.

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Understanding the necessary conditions for undirected graphs to contain a subdivision of a clique is a very natural and well-studied question. One of the most important examples is the following result by Mader [6]:

Theorem 1 ([6])**.** *For every $k \geq 1$, there exists an integer $f(k)$ such that every undirected graph with minimum degree greater than $f(k)$ contains a subdivision of K_k.*

Bollobás and Thomason [1] as well as Komlós and Szemerédi [4] proved that $f(k) = O(k^2)$. In the case of digraphs, there exist examples of digraphs with large out- and indegree without a subdivision of the complete digraph on three vertices, as shown by Thomassen [7] (see also [3] for a simpler construction). However Mader [5] conjectured that an analogue should hold for transitive tournaments TT_k in digraphs with large minimum outdegree.

Conjecture 2 ([5])**.** For every $k \geq 1$, there exists an integer $g(k)$ such that every simple digraph with minimum outdegree greater than $g(k)$ contains a subdivision of TT_k.

The question turned out to be much more difficult than the undirected case, as the existence of $g(5)$ remains unknown. Weakening the statement, Devos, McDonald, Mohar and Scheide [3] made the following conjecture replacing subdivision with immersion and proved it for the case of Eulerian digraphs.

Conjecture 3 ([3])**.** For every $k \geq 1$, there exists an integer $h(k)$ such that every simple digraph with minimum outdegree greater than $h(k)$ contains an immersion of TT_k.

As with subdivisions, Devos et al. showed in [3] the existence of digraphs with large out- and indegree without an immersion of the complete digraph on three vertices. Finding the right value for $h(k)$ in the case of undericted graphs is an interesting question on its own (see [2] for more details).

The goal of this note is to present a proof of this conjecture. Let $F(k,l)$ be the digraph consisting of k vertices $x_1, \ldots, x_k$ and l arcs from x_i to x_{i+1} for every $1 \leq i \leq k-1$. It is clear that $F(k, \binom{k}{2})$ contains an immersion of TT_k, so the following theorem implies Conjecture 3.

Theorem 4. *For every $k \geq 1$ and l, there exists a function $f(k,l)$ such that every digraph with minimum outdegree greater than $f(k,l)$ and multiplicity at most kl contains an immersion of $F(k,l)$.*

Proof. We prove the result for $f(k,l) = 2k^3l^2$ and $l \geq 2$. We proceed by induction on k. For $k = 1$ this is trivial because $F(1,l)$ is one vertex. Suppose now that the result holds for k and assume for a contradiction that it does not hold for $k+1$. Let D be the digraph with the smallest number of arcs and vertices such that D has multiplicity at most $(k+1)l$, all but at most $c_1 = k + (k+1)l$ vertices have outdegree at least $f(k+1,l)$ and without an immersion of $F(k+1,l)$. By minimality of D, every vertex has outdegree exactly $f(k+1,l)$, expect c_1 of them with outdegree 0. Call T the set of vertices of outdegree 0. Suppose we want to remove arcs from D such that the multiplicity of the remaining digraph is at most kl, while keeping the minimum outdegree as large as possible. For a vertex v, the worst case is when, for every vertex $y \in N^+(v)$, the multiplicity of (v,y) is equal to $(k+1)l$. In this case we have to remove at most l arcs for each of the $\frac{f(k+1,l)}{(k+1)l}$ vertices of $N^+(v)$. Therefore, removing T and some of the parallel arcs, we obtain a digraph of outdegree greater than $d' = f(k+1,l) - c_1(k+1)l - \frac{f(k+1,l)}{k+1}$ with multiplicity kl. Because $f(k+1,l) - f(k,l) = 2(3k^2+3k+1)l^2$ and $c_1(k+1)l + \frac{f(k+1,l)}{(k+1)} = k(k+1)l + 3(k+1)^2l^2$, we get that $d' \geq f(k,l)$ and by induction there exists an immersion of $F(k,l)$ in $D-T$. Call $X=\{x_1,\cdots,x_k\}$ the set of vertices of the immersion and, numbering the paths arbitrarily, $P_{i,j}$ the jth directed path of this immersion from x_i to x_{i+1}. We can assume this immersion is of minimum size, so that every vertex in $P_{i,j}$ has exactly one outgoing arc in $P_{i,j}$. Let D' be the digraph obtained from D by removing all the arcs of the $P_{i,j}$ and the vertices $x_1,\dots,x_{k-1}$. By the previous remark, the outdegree of each vertex in D' is either 0 if this vertex belongs to T or at least $f(k+1,l)-(k-1)l-(k-1)(k+1)l$.

For every vertex $y \in D'-x_k$, there do not exist l arc-disjoint directed paths from x_k to y in D', for otherwise there would be an immersion of $F(k+1,l)$ in D. Hence, by Menger's Theorem there exists a set E_y of less than l arcs such that there is no directed path from x_k to y in $D' \setminus E_y$. Define C_y for every vertex $y \in D'-x_k$ as the set of vertices which can reach y in $D' \setminus E_y$. Now take Y a minimal set such that $\cup_{y\in Y} C_y$ covers $D'-x_k$. We claim that Y consists of at least $c_2 \geq \frac{f(k+1,l)-(k-1)l-(k-1)(k+1)l}{l} \geq 2c_1$ elements, as $\cup_{y\in Y} E_y$ must contain all the arcs of D' with x_k as tail.

For each $y \in Y$, define S_y as the set of vertices which belong to C_y and no other $C_{y'}$ for $y' \in Y$. Since Y is minimal, every S_y is non-empty. Note that for $u \in S_y$, if there exists $y' \in Y \setminus y$ and $v \in C_{y'}$ such that $uv \in A(D)$, then $uv \in E_{y'}$. Note that $T \subset Y$ as vertices in T have outdegree 0 and if $y \in Y \setminus T$ then S_y consists only of vertices of outdegree $f(k+1,l)$ in D.

Let R be the digraph with vertex set Y and arcs from y to y' if there is an arc from S_y to $C_{y'}$. As noted before, $d^-_R(y) \leq |E_y| \leq l$. The average outdegree of the vertices of $Y \setminus T$ in R is then at most $\frac{c_1 l+(c_2-c_1)l}{c_2-c_1} \leq 2l$. Let y be a vertex of $R \setminus T$ with outdegree at most this average. Let H be the digraph induced on D' by the vertices in S_y to which we add X, all the arcs that existed in D (with multiplicity) from vertices of S_y to vertices of X and the following arcs: For each $P_{i,j}$, let $z_1,z_2,\dots,z_l = P_{i,j} \cap S_y$, where z_i appears before z_{i+1} on $P_{i,j}$ and add all the arcs (z_i,z_{i+1}) to H. Note that, if (x,y) is an arc of D', then by minimality of the immersion of $F(k,l)$, every time x appears before y on some $P_{i,j}$, then $P_{i,j}$ uses one of the arcs (x,y). Thus for each pair of vertices x and y in H, either $(x,y) \in A(D)$ and the number of (x,y) arcs in H is equal to the one in D, or $(x,y) \notin A(D)$ and the number of (x,y) arcs in H is bounded by $(k-1)l$. This implies that H has multiplicity at most $(k+1)l$.

Claim 4.1. *H is a digraph with multiplicity at most $(k+1)l$, such that all but at most c_1 vertices have outdegree greater than $f(k+1,l)$ and H does not contain an immersion of $F(k+1,l)$.*

Proof of the claim. Suppose H contains an immersion of $F(k+1,l)$, then by replacing the new arcs by the corresponding directed paths along the $P_{i,j}$ we get an immersion of $F(k+1,l)$ in D. Moreover, we claim that the number of vertices in H with outdegree smaller than $f(k+1,l)$ is at most $k+2l+(k-1)l=c_1$. Indeed, the vertices of H that can have outdegree smaller in H than in D are the x_i, or the vertices with outgoing arcs in $E_{y'}$ for some $y' \in Y \setminus y$, or the vertices along the $P_{i,j}$. But with the additions of the new arcs, we know that there is at most one vertex per path $P_{i,j}$ that loses some outdegree in H. ◇

However, since H is strictly smaller than D, we reach a contradiction.

□

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